53–5

Proceedings of the American Academy of Arts and Sciences.

Vol. 53. No. 5.—March, 1918.

THE DYADICS WHICH OCCUR IN A POINT SPACE OF THREE DIMENSIONS.

By C. L. E. Moore and H. B. Phillips.

(Continued from page 3 of cover.)

VOLUME 53.

Proceedings of the American Academy of Arts and Sciences.

Vol. 53. No. 5.— March, 1918.

THE DYADICS WHICH OCCUR IN A POINT SPACE OF THREE DIMENSIONS.

By C. L. E. Moore and H. B. Phillips.

THE DYADICS WHICH OCCUR IN A POINT SPACE OF THREE DIMENSIONS.

By C. L. E. Moore and H. B. Phillips.

Received, November 2, 1917.

In his Ausdehnungslehre, Grassmann gave a discussion of linear transformations of space in which he considered each transformation as determined by a Brucke,[1] or fraction. By using products which he called indeterminate,[2] Gibbs showed that the transformations could be determined by means of bilinear forms called dyadics. These were applied to the study of linear vector functions in three dimensions in the Gibbs-Wilson Vector Analysis. Extensions of this to higher dimensions were given by Gibbs in his lectures on Multiple Algebra an outline of which is contained in an article by E. B. Wilson.[3] H. B. Phillips[4] applied the dyadic to the study of projective transformations in a plane. The fraction of Grassmann does not lend itself readily to algebraical manipulation. This is remedied by the dyadic of Gibbs. The symbol ϕ used by Gibbs does not, however, suggest the nature of the particular dyadic or the invariants and covariants determined from it. In the paper of Phillips a symbolic notation was introduced by which the dyadic appears as the product of a single pair of letters from which invariants and covariants and combinations with other dyadics are obtained by processes of multiplication analogous to the Grassmann products of space elements.

In this paper we have given an exposition of the symbolic notation and have used it to discuss with some completeness the various dyadics occurring in a point space of three dimensions. To aid in the understanding of this we first develop the elements of Grassmann's analysis

---

1 Ausdehnungslehre, 1862, page 240. A good exposition of this is found in Whitehead's Universal algebra, Chapter VI. Book IV.

2 On Multiple Algebra, an address before the section of mathematics and physics of the American Association for the Advancement of Science, by the Vice-President. Proceedings of the American Association for the Advancement of science, **35**. This address is reprinted in the Scientific Papers, **2**.

3 On the theory of double products and strains in hyperspace. Transactions of the Connecticut Academy of Arts and Sciences, **14**, 1908.

4 Some invariants and covariants of ternary collineations, American Journal of Mathematics, **36**, 1914.

in three dimensions and apply the theory to the study of the complex line. The dyadics are of four main types: (a) Those which transform points into points or planes into planes, (b) those which transform points into planes or planes into points, (c) those which transform lines or complexes into lines or complexes, (d) those which transform points or planes into lines or complexes and those which transform lines or complexes into points or planes. The first class represent the collineations and to this type belong most of the dyadics hitherto discussed. The second class represent the correlations. The last two types so far as we know have not been discussed before. These are not in general contact transformations.

By means of double multiplication of two dyadics (one of which may represent an identical transformation) we determine many invariants and covariants. From the geometrical interpretation of this double product we obtain a series of descriptive theorems analogous to the Pappus theorem for the hexagon inscribed in a plane two-line.

INTRODUCTION.

I. Matrices and Outer Products.

1. **Progressive Matrices.** In a former paper[5] we gave an interpretation of the products of Grassmann[6] in which we represented points, lines, etc. as rectangular matrices and expressed the products as operations performed on those matrices. As those products form a fundamental part of the present paper, we shall here briefly outline the method there used.

Our space is a projective point space of three dimensions and so we represent a point A by the matrix

$$A = \| a_1 \, a_2 \, a_3 \, a_4 \| \equiv \| a_i \| \tag{1}$$

where a_1, a_2, a_3, a_4 are the homogeneous coordinates of the point. Two matrices of this kind will be called equal when their corresponding

---

5 A theory of linear distance and angle, These Proceedings, **48**, 1912.

6 Expositions of Grassmann's product theory can be found in the following places:

Ausdehnungslehre, 1862.

Whitehead's Universal Algebra, Chapter I, Book IV.

Encyclopedia, French edition, Complex Number, tome 1, **1**.

The treatment here given is somewhat different from any of the above.

elements are equal. The matrix will be said to be zero only when all its elements are zero.

If two points A and B have coordinates a_i and b_i respectively and $a_i = k\, b_i$, we shall write

$$A = kB.$$

Geometrically A and B are the same point. *But in Grassmannian analysis a point has magnitude as well as position.* The magnitude of A is k times that of B. In this paper we have no need to define unit magnitudes and therefore have not done so.

A linear function of two or more points A, B, C etc. is defined by the matrix

$$\lambda A + \mu B + \nu C + \ldots = \| \lambda a_i + \mu b_i + \nu c_i + \ldots \| \tag{2}$$

If the matrix does not vanish identically, it represents a point in the space determined by $A, B, C,$ etc. Conversely, any point in that space can be represented in that way. For example any point on a line can be represented as a linear function of two, any point in a plane as a linear function of three not on a line, and any point in space as a linear function of four not in a plane. If the matrix vanishes identically, and the multipliers λ, μ, ν, etc. are not all zero, those points lie in a space of lower dimensions than that determined by a like number of independent points.

The Plücker coordinates [7] of a line are certain two rowed determinants

$$p_{ik} = \begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}$$

in the matrix

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{Vmatrix} \tag{3}$$

where a_i and b_i are the coordinates of any two distinct points A and B on the line. We shall represent the line AB by the one-rowed matrix

$$[AB] = \| p_{12}\ p_{13}\ p_{14}\ p_{23}\ p_{24}\ p_{34} \|. \tag{4}$$

As above the matrix will be considered zero if all its elements are zero,

7 The facts concerning Plücker coordinates and line geometry in general which are here assumed are all to be found in Jessop's Line Complex, Cambridge, 1902.

that is, if the points A and B coincide. One matrix is a multiple of another if corresponding elements are proportional. If P, Q are any two distinct points on the line AB, $[PQ]$ is a multiple of $[AB]$. For numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ can then be found such that

$$P = \lambda_1 A + \lambda_2 B,$$
$$Q = \mu_1 A + \mu_2 B.$$

From these equations, by use of (2) it is easy to show that

$$[PQ] = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} [AB].$$

Thus a matrix in addition to representing a line has a definite magnitude.

The sum of two matrices $[AB]$ and $[CD]$ is the matrix each element of which is the sum of corresponding elements of $[AB]$ and $[CD]$. In general the elements of this sum are not the two-rowed determinants of a matrix of the type (3), just as the sum of corresponding Plücker coordinates of two lines are not in general coordinates of a line. A matrix

$$\| c_{12} \; c_{13} \; c_{14} \; c_{23} \; c_{24} \; c_{34} \| \equiv \| C_{ik} \| \tag{5}$$

whose elements c_{ik} are the determinants of (3) will be called simple. Such a matrix represents a line. If, however, the elements c_{ik} cannot be so represented, the matrix will be called complex, we shall say that this represents a *complex line*. The relation of this to the linear complex will be shown later (§5). In what follows the word line will always mean simple line.

If the lines AB and CD intersect in a point P, we can find points Q and R on those lines and assign to them such magnitudes that

$$[AB] = [PQ], \qquad [CD] = [PR].$$

Then

$$\begin{aligned}[AB] + [CD] &= [PQ] + [PR] \\ &= \| p_i(q_k + r_k) - p_k(q_i + r_i) \| = [P(Q + R)].\end{aligned} \tag{6}$$

Therefore *the sum of two intersecting lines is a line.*

If the points A, B, C are not collinear, the coordinates of the plane ABC are the three-rowed determinants of the matrix.

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\|. \tag{7}$$

We shall represent the plane by the matrix

$$[ABC] = \| \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \| \tag{8}$$

where α_i is the coefficient of x_i in the expansion of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix}.$$

Linear functions of planes are defined as in the case of points. If P, Q, R are any three points in the plane ABC, it can be shown as in the case of a line that a number λ can be found such that

$$[PQR] = \lambda[ABC].$$

Thus $[ABC]$ *in addition to representing a plane has a magnitude.*

From four points A, B, C, D we can form a four-rowed matrix. Since this matrix contains only one four-rowed determinant, we shall consider it as a determinant

$$(ABCD) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

Such a matrix of one element we shall consider as a number and indicate this by the use of the parenthesis in the symbol $(ABCD)$.

2. **Progressive Products.** The most fundamental law of multiplication is the distributive law which can be stated in the two forms

$$(A + B)C = AC + BC,$$
$$A(B + C) = AB + AC.$$

The matrix $[AB]$ has this property. For as in equation (6) it is easy to show that

$$[(A + B)C] = [AC] + [BC], \tag{9}$$
$$[A(B + C)] = [AB] + [AC]. \tag{10}$$

Hence we consider $[AB]$ as a product of A and B. The process of multiplication consists in placing the matrix A over the matrix B to form the two-rowed matrix

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{Vmatrix}$$

and from this determining the elements of the matrix $[AB]$. From the definition it is clear that

$$[AB] = -[BA], \tag{11}$$
$$[AA] = 0. \tag{12}$$

Similarly the matrix $[ABC]$ can be regarded as a product of $[AB]$ and C, of A and $[BC]$ or of A, B, and C, the process of multiplication consisting always in forming the matrix

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\|$$

and determining the matrix $[ABC]$ by using the determinants in this as elements. From this definition it is evident that

$$\begin{aligned} [A \cdot BC] &= [AB \cdot C] = [ABC], \\ [AB(C + D)] &= [ABC] + [ABD], \\ [ABC] &= -[ACB] = [CAB]. \end{aligned}$$

The dot is used throughout our work to show the order of operations. Thus

$$[A \cdot BC] = [A[BC]].$$

Products involving the complex line will be considered in §5.

Since the coordinates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of a plane can have arbitrary values, not all zero, a linear function of planes is a plane unless it vanishes identically.

The quantity $(ABCD)$ can be regarded as a product of A and $[BCD]$, of $[AB]$ and $[CD]$, of A, B, C and D etc. Hence by definition and the properties of determinants.

$$(ABCD) = (A \cdot BCD) = (AB \cdot CD) = -(ABDC), \text{ etc.}$$

It is to be noted that the sign of $(ABCD)$ is changed when two of the points A, B, C, D are interchanged. These products are called progressive because every additional factor increases the dimension of the product.

If any of these products vanish it shows that the points lie in a space of lower dimension than is determined by a like number of independent points. For example if

$$[ABC] = 0$$

then A, B, C lie on a line as can easily be verified by expressing the matrix in terms of coordinates.

3. **Regressive Matrices and products.** We can consider space as generated by planes as well as by points. If its coordinates are α_i, a plane α is represented by the matrix

$$\alpha = \| \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \|.$$

The same plane may be represented by the matrix $[ABC]$ of any three noncollinear points lying in it. If α_i is equal to the coefficient of x_i in the expansion of the determinant $(ABCX)$, we shall write

$$\alpha = [ABC].$$

The line of intersection of two planes α and β can be represented by the matrix

$$[\alpha\beta] = \| q_{34} \ q_{42} \ q_{23} \ q_{14} \ q_{31} \ q_{12} \|, \tag{13}$$

where

$$q_{ik} = \begin{vmatrix} \alpha_i & \alpha_k \\ \beta_i & \beta_k \end{vmatrix}.$$

If the same line is the join of two points A and B, we know from line geometry that the coordinates q_{ik} are proportional to the coefficients of the minors $\begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$ in the determinant $(ABxy)$. If q_{ik} is *equal* to the coefficient of $| \, x_i \ y_k \, |$ in that determinant we shall write

$$[\alpha\beta] = [AB].$$

Three planes α, β, γ intersect in a point A. The coordinates a_i of this point are proportional to the coefficients of ξ_i in the determinant [8] $(\xi\alpha\beta\gamma)$. In particular, if a_i is equal to the coefficient of ξ_i in that determinant, we write

$$A = [\alpha\beta\gamma].$$

There is a determinant $(\alpha\beta\gamma\delta)$ of four planes just as of four points. The matrices $[\alpha\beta]$, $[\alpha\beta\gamma]$, $[\alpha\beta\gamma\delta]$ can be regarded as products. They

---

[8] It is to be observed that the variable ξ is put in the first row of this determinant while in §1 we wrote for points $(ABCX)$. This change is made in order to make the reduction formulas agree in sign with the ones Grassmann gave.

obey the same laws as the corresponding products of points. These products of matrices expressed in plane coordinates we shall call *regressive* because each additional factor decreases the dimension of the product.

If a regressive product vanishes it shows that the planes determine a space of higher dimension than a like number of independent planes determine. For example if

$$(\alpha\beta\gamma\delta) = 0$$

the four planes $\alpha, \beta, \gamma, \delta$ intersect in a point as is easily shown by writing the matrix in terms of coordinates.

4. **Mixed products and reduction formulas.** If the total number of points in a set of progressive matrices is equal to or less than four, the matrices are multiplied together as already explained. If the total number of planes in a set of regressive matrices is equal to or less than four, they are multiplied together in a similar manner. In both cases the product is associative. If the total number of points or planes in two matrices is greater than four,[9] we have not defined the product. In that case we replace each of the matrices by its equivalent in complementary elements. We shall say that points and planes are complementary elements and that lines are complementary to lines. For example, we could replace $[AB]$ by $[\alpha\beta]$ where

$$[AB] = [\alpha\beta],$$

$[ABC]$ by α where

$$[ABC] = \alpha,$$

etc. The total number of elements (points or planes) in the new matrices will then be less than four and the product can be formed as before. If the total number of elements is equal to four, the product will be the same whether the matrices are expressed in points or planes. If the matrices are of different kinds (one progressive, the other regressive) we express one of them by its complementary form. Thus in every case of the product of two factors there is a definite result that has a meaning. We call this the *outer* product of the two factors.

The product of a line $[AB]$ and a plane γ is the point of intersection of the line and plane considered as having a certain magnitude. For,

9 It is assumed here that the number of elements (points or planes) in either factor is not greater than four.

to obtain this product we replace $[AB]$ by its complementary form $[\alpha\beta]$. Then

$$[AB]\cdot\gamma = [\alpha\beta\gamma].$$

The result is the point in which α, β, γ intersect. Since α and β are planes passing through $[AB]$, this is the point in which $[AB]$ intersects γ.

This method of obtaining the product is simple in conception but not analytically convenient. We shall therefore give a set of reduction formulas by which the same results can be obtained in more useful form. The proof of these formulas is not essential for our purposes. Hence we give them without proof.[10] The roman letters represent points, the greek letters planes.

$$\begin{aligned}
[ABC\cdot DEF] &= (ABCF)[DE] - (ABCE)[DF] + (ABCD)[EF] \\
&= (ADEF)[BC] - (BDEF)[AC] + (CDEF)[AB]. && (14)\\
[ABC\cdot DE] &= (ABCE)D - (ABCD)E \\
&= (ABDE)C - (ACDE)B + (BCDE)A. && (15)\\
[\alpha\cdot ABC] &= (\alpha C)[AB] - (\alpha B)[AC] + (\alpha A)[BC]. && (16)\\
[\alpha\cdot AB] &= (\alpha B)A - (\alpha A)B. && (17)
\end{aligned}$$

Similar formulas can be obtained by replacing points by planes and planes by points. The following formula

$$(ABCD)E = (ABCE)D - (ABDE)C + (ACDE)B - (BCDE)A \quad (18)$$

and the one that results by replacing points by planes are also sometimes found useful.

5. **The complex.** Let A, B, C, D be four points not in a plane. Any two points P_1, P_2 of space can be expressed as linear functions of A, B, C, D.

$$P_1 = \lambda_1 A + \mu_1 B + \sigma_1 C + \rho_1 D,$$
$$P_2 = \lambda_2 A + \mu_2 B + \sigma_2 C + \rho_2 D.$$

Hence

$$\begin{aligned}
[P_1P_2] &= [(\lambda_1 A + \mu_1 B + \sigma_1 C + \rho_1 D)(\lambda_2 A + \mu_2 B + \sigma_2 C + \rho_2 D)] \\
&= (\lambda_1\mu_2 - \lambda_2\mu_1)[AB] + (\lambda_1\sigma_2 - \lambda_2\sigma_1)[AC] + [\lambda_1\rho_2 - \lambda_2\rho_1)[AD] \\
&\quad + (\mu_1\sigma_2 - \mu_2\sigma_1)[BC] + (\mu_1\rho_2 - \mu_2\rho_1)[BD] + (\sigma_1\rho_2 - \sigma_2\rho_1)[CD].
\end{aligned}$$

This shows that any line is a linear function of the six edges of a tetrahedron. The sum of any number of lines is then obviously a linear

10 See Linear distance and angle or any of the works mentioned in note 6.

function of the six edges. These edges can be divided into two sets, those through the point A and those in the plane $[BCD]$. The lines through a point all intersect and so their sum is a line. For the same reason the sum of any number of lines in a plane is a line. Therefore *the sum of any number of lines can be expressed as a sum of two lines, one through an arbitrary point A, the other in an arbitrary plane $[BCD]$ not passing through A.*

We have said that a matrix

$$p = \| c_{12} \; c_{13} \; c_{14} \; c_{23} \; c_{24} \; c_{34} \|$$

represents a complex line if the elements c_{ik} are not the Plücker coordinates of a line. It can be expressed as the sum of six matrices

$$\| c_{12} \; 0 \; 0 \; 0 \; 0 \; 0 \|, \quad \| 0 \; c_{13} \; 0 \; 0 \; 0 \; 0 \|, \text{ etc.}$$

Each of these represents a line. Hence *any complex line can be represented as the sum of six lines and consequently as the sum of two lines.*

Any complex p can then be expressed in the form

$$p = [AB] + [CD].$$

By the product of p and any element Γ (point, line, plane or complex) we mean the sum

$$[p\Gamma] = [AB\Gamma] + [CD\Gamma].$$

If p is expressed in a different form

$$p = [A'B'] + [C'D']$$

it is clear that $[p\Gamma]$ will have the same value as before. For example, if Γ is a point the coordinates of the plane $[AB\Gamma] + [CD\Gamma]$ are definite linear functions of the sums

$$\begin{vmatrix} a_i & b_i \\ a_k & b_k \end{vmatrix} + \begin{vmatrix} c_i & d_i \\ c_k & d_k \end{vmatrix}$$

and the coordinates of $[A'B'\Gamma] + [C'D'\Gamma]$ are the same functions of the sums

$$\begin{vmatrix} a_i' & b_i' \\ a_k' & b_k' \end{vmatrix} + \begin{vmatrix} c_i' & d_i' \\ c_k' & d_k' \end{vmatrix}.$$

Since $[AB] + [CD] = [A'B'] + [C'D']$ those sums are by definition equal.

If the sum of $[AB]$ and $[CD]$ is a line, AB and CD intersect. For, if

$$[AB] + [CD] = [PQ],$$

then

$$[ABP] + [CDP] = [PQP] = 0.$$

This shows that $[ABP]$ and $[CDP]$ are the same plane and hence AB and CD lie in a plane and so intersect.

The product of p by itself is

$$[pp] = [(AB + CD)(AB + CD)] = 2(AB\,CD).$$

If p is a complex line $[AB]$ and $[CD]$ do not intersect and so $(AB\,CD)$ is not zero. If, however, p is a line $[AB]$

$$(pp) = (AB \cdot AB) = 0. \tag{19}$$

Hence *the necessary and sufficient condition that a matrix*

$$p = \|\, c_{ik} \,\|$$

represent a simple line is

$$(pp) = 0.$$

Let

$$p = [AB] + [CD]$$

be a complex line and

$$l = [XY]$$

be a line. The equation

$$(pl) = (ABXY) + (CDXY) = 0$$

is a linear equation in the coordinates

$$\begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$$

of the line l. Hence the lines satisfying this equation constitute a linear complex. This complex is the totality of lines l satisfying the equation

$$(pl) = 0. \tag{20}$$

This is a different thing from p which in a sense is the envelope of the lines just as a point in a plane is different from the set of lines passing through it. For this reason we call p a complex line to distinguish it

from a linear complex. Where no ambiguity results we shall use the word complex for either the complex line or the linear complex.

A complex p can be represented as a linear function of two lines of which one, l, can be any line not belonging to the linear complex,

$$(pl) = 0.$$

For a number λ can be found satisfying the equation.

$$[(p - \lambda l)(p - \lambda l)] = (pp) - 2\lambda(pl) + \lambda^2(ll) = 0. \tag{21}$$

Since l is a line, by (19), $(ll) = 0$. Also by assumption (pl) is not zero. Hence if

$$\lambda = \frac{(pp)}{(pl)},$$

$p - \lambda l$ is a line l' and so

$$p = \lambda l + l'.$$

The two lines l and l' are said to be *polar with respect to the complex.* Any line of the complex that intersects one of them will intersect the other also. For, if q is a line of the complex cutting l,

$$(pq) = 0, \qquad (lq) = 0.$$

Hence from the equation

$$(pq) = \lambda(lq) + (l'q)$$

it is seen that $(l'q) = 0$ and so q intersects l.

Let P be any point and

$$p = [AB] + [CD]$$

a complex. Then

$$[Pp] = [PAB] + [PCD] \tag{22}$$

is a plane. If Q is any point in this plane

$$(PpQ) = (p \cdot PQ) = 0.$$

Hence $[PQ]$ is a line of the complex. All lines passing through P and lying in the plane $[Pp]$ therefore belong to the complex. Hence $[Pp]$ is what is known as the *polar plane of P with respect to the complex.* Similarly, if α is any plane

$$[\alpha p]$$

is what is called the pole of α with respect to the complex. All the lines of the complex, lying in the plane α, pass through the point $[\alpha p]$.

If l and l' are polar lines with respect to p,

$$p = \lambda l + \mu l'.$$

If then X is a point on l, $[Xl] = 0$ and so

$$[Xp] = \mu[Xl']$$

which shows that the polar plane of X with respect to p contains the line l'.

If the product (pq) of two complexes p and q is zero, the complexes are said to be in involution. In this case we shall say that the two complexes intersect from analogy with the case of two lines which intersect if their product is zero.

If two lines l_1, l_2 satisfy a linear relation

$$\lambda_1 l_1 + \lambda_2 l_2 = 0, \qquad \lambda_1, \lambda_2 \neq 0$$

they coincide in position. If three lines satisfy a linear relation

$$\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = 0 \qquad \lambda_1, \lambda_2, \text{ or } \lambda_3 \neq 0$$

any line cutting two of them cuts the third and so they belong to a plane pencil. If four lines satisfy a linear relation

$$\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 + \lambda_4 l_4 = 0 \qquad \lambda_1 \lambda_2, \lambda_3, \text{ or } \lambda_4 \neq 0$$

any line cutting three cuts the fourth also. Hence they belong to the same system of generators on a quadric. If five lines satisfy a linear relation, the two lines cutting four of them will cut the fifth also and so they belong to a linear congruence. If six lines satisfy a linear relation they belong to a linear complex.

Similarly if two complex lines satisfy a linear relation the linear complexes represented by them are identical.

If three complex lines satisfy a linear relation, the linear complexes have a common congruence. If four complex lines satisfy a linear relation, the complexes have one system of generators on a quadric surface in common. If five satisfy a linear relation, the complexes have two lines in common. If six complex lines satisfy a linear relation the complexes are in involution with a fixed complex.

II. Dyadics.

6. **The indeterminate product.** Grassmann [11] showed that there are four kinds of products characterized by laws which are the same for units and for any linear functions of the units. These are the algebraic, $AB = BA$, combinatory $AB = -BA$, that in which all products are zero, and that in which there is no relation between the products of independent units. Grassmann discussed the first two in detail, but it was left to the genius of Willard Gibbs to recognize the importance of the last. Because of the indefinite character of the result he called it the *indeterminate product.*

We represent the indeterminate product of A and B by the notation AB. By definition this product obeys the following laws.

$$\begin{aligned} AB + CD &= CD + AB, \\ (AB + CD) + EF &= AB + (CD + EF), \\ \lambda AB + \mu AB &= (\lambda + \mu)AB, \\ A(B + C) &= AB + AC, \\ (A + B)C &= AC + BC, \\ \lambda AB = (\lambda A)B &= A(\lambda B), \\ 0 \cdot AB &= 0, \end{aligned}$$

where A and B are extensive quantities (points, lines, planes or complexes) and λ, μ numbers. If the factors A and B of AB are replaced by equivalent expressions, and the product expanded by the above laws the sum of terms obtained is said to be equal to AB.

Gibbs called the product AB a *dyad.* If $A_1, A_2, A_3, \ldots . A_n$ are extensive quantities of the same kind (points, lines, complexes, or planes) and $B_1, B_2, B_3, \ldots B_n$ extensive quantities of the same kind, the sum

$$\Phi = A_1B_1 + A_2B_2 + \ldots . A_nB_n$$

is called a *dyadic.* The A's are called the antecedents and the B's the consequents of the dyadic.

There are two products of a dyad AB and an extensive quantity C. These are

$$\begin{aligned} AB \cdot C &= A[BC], \\ C \cdot AB &= [CA]B. \end{aligned}$$

---

11 Ausdehnungslehre, chapter 2, page 33.

Similarly there are two products $C\Phi$ and ΦC of C and a dyadic Φ. These are obtained by multiplying C and each dyad of Φ as above and adding the results.

If X is complementary to the consequents

$$\Phi X = A_1(B_1X) + A_2(B_2X) + \dots + A_n(B_nX),$$

is an extensive quantity of the same kind (dimensions) as the antecedents. Hence

$$Y = \Phi X$$

is a linear transformation in which to each element X corresponds an element Y of the same kind as the antecedents. It can be shown that the most general linear transformation of these elements can be expressed in this way. Similarly if X is complementary to the antecedents

$$Y = X\Phi$$

is a transformation of elements X into elements Y of the same kind as the consequents.

In this paper we shall consider the following types of dyadics.

(a) The one-one dyadic, in which the antecedents and consequents are both points.

(b) The three-three dyadic, in which the antecedents and consequents are both planes.

(c) The one-three dyadics, in which the antecedents are points and the consequents are planes.

(d) The two-two dyadics, in which the antecedents and consequents are both lines or complexes.

(e) The one-two and two-one dyadics.

(f) The two-three and three-two dyadics.

7. **Idemfactors.** A dyadic I such that

$$X = IX \tag{24}$$

for all elements X of a given kind, is called an *idemfactor*. In this case the antecedents and consequents must be complementary in kind. Therefore there are three idemfactors, a one-three, a three-one and a two-two.

There is only one idemfactor of each of these types, For, if I_1 and I_2 are dyadics such that

$$X = I_1X, \qquad X = I_2X,$$

then

$$(I_1 - I_2)X = 0.$$

Since this is true for all elements X of the same kind, that is, of the same dimension it is easy to show that

$$I_1 - I_2 = 0,$$

which shows that I_1 and I_2 are identical.

If ξ is complementary to X, the idemfactor I in (24) satisfies the equation

$$\xi = \xi I.$$

For, if we multiply each side of (24) by ξ we get

$$(\xi X) = (\xi \cdot IX) = (\xi I \cdot X).$$

Hence

$$[(\xi - \xi I)X] = 0$$

for all elements X and so

$$\xi - \xi I = 0.$$

8. **Conjugate, self-conjugate and anti-self-conjugate dyadics.** The dyadic Φ_c obtained by interchanging the antecedents and consequents of Φ is called the *conjugate* of Φ. Thus if

$$\Phi = A_1B_1 + A_2B_2 + \ldots . + A_nB_n,$$
$$\Phi_c = B_1A_1 + B_2A_2 + \ldots . + B_nA_n.$$

If X is complementary to the consequents, it is clear that

$$\Phi X = \pm X\Phi_c.$$

The sign is plus or minus according as $[XB_i]$ is equal to $[B_iX]$ or to its negative. Similarly if γ is complementary to the antecedents

$$\gamma\Phi = \pm \Phi_c\gamma.$$

A dyadic Φ is called self-conjugate if

$$\Phi = \Phi_c$$

and anti-self-conjugate if

$$\Phi = -\Phi_c.$$

In each case the antecedents and consequents must be quantities of the same kind. *Any dyadic whose antecedents and consequents are quantities of the same kind can be expressed as the sum of a self-conjugate and of an anti-self-conjugate dyadic.* For.

$$\Phi = \tfrac{1}{2}(\Phi + \Phi_c) + \tfrac{1}{2}(\Phi - \Phi_c)$$

and it is clear that $\frac{1}{2}(\Phi + \Phi_c)$ is self-conjugate and $\frac{1}{2}(\Phi - \Phi_c)$ anti-self-conjugate.

The one-three idemfactor is minus the conjugate of the three-one and vice versa, because if Φ is the three-one idemfactor $\alpha\Phi = \alpha$, or $\alpha = -\Phi_c\alpha$, since $[\alpha A_i] = -[A_i\alpha]$. The two-two idemfactor is self-conjugate.

9. **Products of two dyadics.** By the product $AB \cdot CD$ of two dyads AB and CD is meant the indeterminate product $A[BC]D$ obtained by taking the outer product (see §4) of the adjacent factors B, C. In case $[BC]$ is not a number, the result is an indeterminate product of three factors or a triad. If $[BC]$ is a number, it is commutative with A and the result is the dyad $(BC)AD$. Similarly the product of three dyads AB, CD, EF is defined as

$$AB \cdot CD \cdot EF = A[BC]\,[DE]F$$

and the product of two dyads and an extensive quantity E is

$$AB \cdot CD \cdot E = A[BC]\,[DE]$$

etc. In each case it is clear that the product is associative so far as dyads or dyads and extensive quantities are concerned.

The product $\Phi\Psi$ of two dyadics is defined as the result of multiplying each dyad of Φ by each dyad of Ψ and adding the results. Similarly the product of three or more dyadics is obtained by multiplying them distributively. Since the product of dyads is associative, the product of dyadics is associative.

We have seen that the dyadic Ψ as an operator transforms extensive quantities X complementary to the consequents into extensive quantities ΨX of the same dimension as the antecedents. *If the consequents of Φ are complementary to the antecedents of Ψ, $\Phi\Psi$ is a dyadic which as an operator is equivalent to the operator Ψ followed by the operator Φ.* For if

$$Y = \Psi X, \qquad Z = \Phi Y,$$

by the associative law

$$Z = \Phi \cdot \Psi X = \Phi\Psi \cdot X.$$

In the same way, if X is complementary to the antecedents of Φ, $X\Phi\Psi$ is equivalent to the transformation $X\Phi$ followed by Ψ as postfactor.

If the consequents of Φ are complementary to the antecedents of an idemfactor I,

$$\Phi I = \Phi.$$

For, if

$$\Phi = A_1B_1 + A_2B_2 + \ldots + A_nB_n$$

by the definition of an idemfactor

$$B_iI = B_i,$$

and so

$$\begin{aligned} \Phi I &= A_1B_1I + A_2B_2I + \ldots + A_nB_nI \\ &= A_1B_1 \ + A_2B_2 \ + \ldots A_nB_n = \Phi. \end{aligned}$$

Similarly if the consequents of I are complementary to the antecedents of Φ,

$$I\Phi = \Phi.$$

If there exists a dyadic Ψ such that

$$\Phi\Psi = I$$

or

$$\Psi\Phi = I$$

Φ and Ψ are said to be inverse dyadics. In many cases these two relations of Ψ and Φ will be equivalent, but there are some cases in which they are not.

10. **Symbolic Notation.** If we write a dyadic as

(A) $$A_1B_1 + A_2B_2 + \ldots + A_nB_n = \Sigma A_iB_i,$$

the products of the dyadic with an extensive quantity X are

(B) $$A_1[B_1X] + A_2[B_2X] + \ldots + A_n[B_nX] = \Sigma A_i[B_iX],$$
(C) $$[XA_1]B_1 + [XA_2]B_2 + \ldots + [XA_n]B_n = \Sigma[XA_i]B_i.$$

Similarly if

(D) $$C_1D_1 + C_2D_2 + \ldots + C_nD_n = \Sigma C_iD_i$$

is a second dyadic, the products of the two dyadics are

(E) $$\Sigma_{ik}A_i[B_iC_k]D_k, \qquad \Sigma_{ik}C_i[D_iA_k]B_k.$$

With a dyadic is associated a number or extensive quantity

(F) $$[A_1B_1] + [A_2B_2] + \ldots + [A_nB_n] = \Sigma[A_iB_i]$$

and the product of this with an extensive quantity a is

(G) $$[A_1B_1 \cdot a] + [A_2B_2 \cdot a] + \ldots + [A_nB_n \cdot a] = \Sigma[A_iB_i \cdot a].$$

Now if we observe the above relations (A) to (G) we will see that a similarity runs throughout which can be made the basis of a symbolic notation. This consists in replacing ΣA_iB_i by a symbolic dyad AB and ΣC_iD_i by a symbolic dyad CD. We then have

$$\begin{aligned}
AB &= \Sigma A_iB_i,\\
A[BX] &= \Sigma A_i[B_iX],\\
[XA]B &= \Sigma[XA_i]B_i,\\
CD &= \Sigma C_iD_i,\\
AB \cdot CD &= \Sigma A_i[B_iC_k]D_k,\\
CD \cdot AB &= \Sigma C_i[D_iA_k]B_k,\\
[AB] &= \Sigma[A_iB_i],\\
[AB \cdot a] &= \Sigma[A_iB_i \cdot a].
\end{aligned}$$

The symbolism consists in each case in omitting the summation sign and the subscripts. Conversely each symbolic expression is equivalent to the non-symbolic form obtained by introducing summation signs and attaching subscripts to the proper letters.

By this notation operations on dyadics appear like operations with simple dyads and the results can be expanded and handled much the same as ordinary extensive quantities. Thus if A_i and B_i are points and a a plane, the last expression can be expanded in the form

$$[AB \cdot a] = \Sigma[A_iB_i \cdot a] = \Sigma(A_ia)B_i - \Sigma(B_ia)A_i,$$

and if we put $\Sigma(A_ia)B_i = (Aa)B$, $\Sigma(B_ia)A_i = (Ba)A$, we have

$$[AB \cdot a] = (Aa)B - (Ba)A,$$

which follows the ordinary Grassmann formula for expansion of a product.

Two dyadics AB and CD have a double product defined by

$$AB{:}CD = [AC]\,[BD].$$

The significance and properties of these double products will be discussed later.

It is clear that these symbolic forms will not be ambiguous if each of the letters A, B, C, D does not occur more than once in a product. If the same dyadic occurs more than once in a product, we represent it in each of its positions by a different pair of letters. Thus to obtain the product of AB with itself, we let $AB = A'B'$ and so write the result in the form

$$A[BA']B' = \Sigma A_i[B_iA_k]B_k$$

If we write it in the form $A[BA]B$ it is not clear whether this means $A[BA']B'$ or $A[B'A']B$. It is evident also that a product containing one of the letters A or B without the other, such as

$$[AC]D$$

has no significance.

11. **The one-three ond three-one dyadics.** These dyadics have been investigated quite extensively by Gibbs,[2] Wilson[3] and Phillips.[4] We shall therefore state only a few facts about them.

A one-three dyadic has the form

$$B\beta = B_1\beta_1 + B_2\beta_2 + \ldots . + B_n\beta_n$$

where the B's are points and the β's planes. Since the planes can be expressed as linear functions of any four not passing through a point, the dyadic can be expressed in the simpler form,

$$B\beta = B_1\beta_1 + B_2\beta_2 + B_3\beta_3 + B_4\beta_4,$$

where β_1, β_2, β_3, β_4 are any four planes not passing through a point. In the same way instead of the four planes, the four points B_1, B_2 B_3, B_4 could be assigned arbitrarily.

As an operator on points X, this dyadic gives a collineation

$$Y = B(\beta X) = B_1(\beta_1 X) + B_2(\beta_2 X) + B_3(\beta_3 X) + B_4(\beta_4 X).$$

If X is the intersection of the planes β_2, β_3, β_4

$$X = [\beta_2\beta_3\beta_4], \qquad (\beta_2 X) = (\beta_3 X) = (\beta_4 X) = 0,$$

and

$$Y = B_1(\beta_1\beta_2\beta_3\beta_4).$$

Hence the vertices of the tetrahedron β_1, β_2, β_3, β_4 pass by the collineation into the points B_1, B_2, B_3, B_4.

Let A_1, A_2, A_3, A_4, be four points, not in a plane, with magnitudes so chosen that

$$(A_1A_2A_3A_4) = 1.$$

Let
$$[A_1A_2A_3] = \alpha_4, \qquad [A_1A_2A_4] = -\alpha_3,$$
$$[A_1A_3A_4] = \alpha_2, \qquad [A_2A_3A_4] = -\alpha_1.$$

It is then easily seen that

$$[\alpha_i\ A_i] = 1, \qquad i = 1,\ 2,\ 3,\ 4 \tag{25}$$
$$[\alpha_i\ A_j] = 0, \qquad i \neq j. \tag{26}$$

Four points A_i and four planes a_i satisfying these equations are said to form a *reciprocal system*.[12]

If the points A_i and the planes α_i form a reciprocal system, the dyadic

$$I = A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + A_4\alpha_4 \tag{27}$$

is an idemfactor. For equations (25), (26) show that

$$IA_i = A_i.$$

Since any point X can be expressed in the form

$$X = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4,$$

it follows that

$$IX = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = X.$$

If the antecedents are points lying in a plane or the consequents are planes passing through a point, the dyadic is called *singular*. Suppose B_1, B_2, B_3, B_4 lie in a plane. Then P_4 can be expressed as a linear function of P_1, P_2, P_3. Hence the dyadic

$$B\beta = B_1\beta_1 + B_2\beta_2 + B_3\beta_3 + B_4\beta_4$$

can be written in the form

$$B\gamma_1 = B_1\gamma_1 + B_2\gamma_2 + B_3\gamma_3. \tag{28}$$

---

12 The reciprocal system was fundamental in Gibbs' work on dyadics. See Gibbs-Wilson Vector Analysis. Also the paper by Wilson mentioned in note 3.

A similar result will be obtained if the planes β_i pass through a point. Hence any singular dyadic can always be expressed as the sum of three dyads. Conversely if the dyadic can be expressed in this form, it is evidently singular.

If X is the point of intersection of the planes $\gamma_1, \gamma_2, \gamma_3$ in (28)

$$B(\beta X) = 0.$$

If a one-three dyadic is singular there is then a point X such that $B\ (\beta X)$ is zero. Conversely, if there is such a point the dyadic is singular. For, if

$$B(\beta X) = B_1(\beta_1 X) + B_2(\beta_2 X) + B_3(\beta_3 X) + B_4(\beta_4 X) = 0$$

either $(\beta_1 X) = (\beta_2 X) = (\beta_3 X) = (\beta_4 X) = 0$ and the four planes pass through the point X, or the four points B_i satisfy a linear relation and so lie in a plane.

If a one-three dyadic Φ is not singular it has an inverse Φ^{-1} such that

$$\Phi\Phi^{-1} = I = \Phi^{-1}\Phi \tag{29}$$

is the one-three idemfactor. To show this, let

$$\Phi = B_1\beta_1 + B_2\beta_2 + B_3\beta_3 + B_4\beta_4.$$

Since B_1, B_2, B_3, B_4 do not lie in a plane, we can associate with them four planes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that the points and planes form a reciprocal system. Similarly let C_1, C_2, C_3, C_4 form with $\beta_1, \beta_2, \beta_3, \beta_4$ a reciprocal system. Then

$$\begin{aligned} (\gamma_i B_i) &= (\beta_i C_i) = 1, \\ (\gamma_i B_j) &= (\beta_i C_j) = 0, \qquad i \neq j. \end{aligned} \tag{30}$$

Using these equations it is easy to show that

$$\Phi^{-1} = C_1\gamma_1 + C_2\gamma_2 + C_3\gamma_3 + C_4\gamma_4$$

has the required property. Also

$$\Phi^{-1}\Phi = I = \Phi\Phi^{-1}. \tag{31}$$

The quantity

$$(B\beta) = (B_1\beta_1) + (B_2\beta_2) + (B_3\beta_3) + (B_4\beta_4)$$

is called the *scalar* of the dyadic. It is evidently independent of the form in which the dyadic is written. For, if the B's and β's are

expressed as linear functions of new elements B'_i and β'_i and so the dyadic is expressed in a new form, by the distributive law of outer multiplication, the scalar of the dyadic will at the same time be transformed into the scalar of the dyadic in the new form. The same conclusion follows also from the fact that the laws of the indeterminate multiplication are included among those of any distributive multiplication whose operations are commutative with multiplication by a scalar. Hence if an equation is satisfied by dyads, the equation will still be satisfied, if the indeterminate products are replaced by any such distributive products.

A function of a dyadic independent of the form in which the dyadic is written will be called an invariant if it is a scalar, a covariant if it is an extensive quantity or dyadic. In a similar way we define invariants and covariants of two or more dyadics.

If the scalar of the dyadic is zero it was shown by Pasch [13] that there exist certain tetrahedra such that each vertex passes, by the collineation, set up by the dyadic, into a point of the opposite plane and, conversely, if any such tetrahedron exists the scalar of the dyadic is zero.

The discussion of the three-one dyadic runs exactly similar to the one-three of which it is the conjugate. As an operator it gives also a collineation but in this case it is a collineation in planes instead of in points. If the antecedents α_i and the consequents A_i form a reciprocal system, the dyadic

$$\alpha A = -[\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4] \tag{32}$$

is the three-one idemfactor.

The scalar of a three-one dyadic is the negative of the scalar of its conjugate (which is one-three) and consequently the vanishing of this scalar has the same signification.

12. **One-one and three-three dyadics.** A one-one dyadic has the form

$$BC = B_1C_1 + B_2C_2 + \ldots + B_nC_n.$$

Since the points $B_1, B_2, \ldots B_n$ can be expressed as linear functions of four, the dyadic can always be reduced to the form

$$BC = B_1C_1 + B_2C_2 + B_3C_3 + B_4C_4. \tag{33}$$

13 Vollkommene Invariante, Math. Ann. Vol. 52, page 128.

Any four points not lying in a plane can be taken as antecedents or as consequents.

As an operator on planes the dyadic gives a correlation which transforms each plane ξ into a point

$$X = B(C\xi) = B_1(C_1\xi) + B_2(C_2\xi) + B_3(C_3\xi) + B_4(C_4\xi).$$

If ξ is the plane $[C_2C_3C_4]$,

$$(B_2\xi) = (B_3\xi) = (B_4\xi) = 0,$$

and

$$X = B_1(C_1C_2C_3C_4).$$

If then the consequents do not lie in a plane, the correlation transforms the planes of the tetrahedron $C_1\,C_2\,C_3\,C_4$ into the points B_1, B_2, B_3, B_4.

Since the antecedents and consequents of the one-one dyadic are extensive quantities of the same dimension it can be expressed as the sum of a self-conjugate and an anti-self-conjugate dyadic.

$$\Phi = \tfrac{1}{2}(\Phi + \Phi_c) + \tfrac{1}{2}(\Phi - \Phi_c). \tag{34}$$

We therefore consider these two types of dyadics first.

A self-conjugate one-one dyadic can be expressed in the form

$$BB = B_1B_1 + B_2B_2 + B_3B_3 + B_4B_4. \tag{35}$$

To show this, express the antecedents and consequents in terms of four points, A_1, A_2, A_3, A_4, not lying in a plane. The dyadic will then take the form

$$\Sigma\lambda_{ik}A_iA_k, \qquad i, k = 1, 2, 3, 4.$$

Since the dyadic is self conjugate

$$\Sigma\lambda_{ik}A_iA_k = \Sigma\lambda_{ik}A_kA_i.$$

Hence

$$\lambda_{ik} = \lambda_{ki}. \tag{A}$$

Consider the quadratic form

$$\Sigma\lambda_{ik}x_ix_k \tag{B}$$

where the x's are numbers. Four linear functions

$$y_n = \Sigma\mu_{nm}x_m \tag{C}$$

can be found such that

$$\Sigma\lambda_{ik}x_ix_k = \Sigma y_n^2. \tag{D}$$

Let

$$B_n = \Sigma \mu_{nm} A_m. \tag{E}$$

Using these values, let

$$\Sigma B_n B_n = \Sigma \nu_{ik} A_i A_k. \tag{F}$$

Since (F) is obtained from (E) in the same way that (D) is from (C) except that $A_i A_k$ is not equal to $A_k A_i$ we must have

$$\nu_{ik} + \nu_{ki} = \lambda_{ik} + \lambda_{ki} = 2\lambda_{ik}. \tag{G}$$

Also, since the left side of (F) is symmetric

$$\nu_{ik} = \nu_{ki}. \tag{H}$$

From (G) and (H) we get

$$\nu_{ik} = \lambda_{ik}.$$

Therefore (F) is equivalent to

$$BB = \Sigma B_n B_n = \Sigma \lambda_{ik} A_i A_k.$$

A self-conjugate dyadic represents a polarity. For the transform of a plane ξ is the point

$$X = B(B\xi) = B_1(B_1\xi) + B_2(B_2\xi) + B_3(B_3\xi) + B_4(B_4\xi). \tag{36}$$

This point is the pole of ξ with respect to the quadric surface

$$(B_1\xi)^2 + (B_2\xi)^2 + (B_3\xi)^2 + (B_4\xi)^2 = 0.$$

The points B_1, B_2, B_3, B_4 form a self-polar tetrahedron with respect to the quadric. For, the pole of the plane $[B_1 B_2 B_3]$, from (36), is

$$B_1(B_1B_1B_2B_3) + B_2(B_2B_1B_2B_3) + B_3(B_3B_1B_2B_3) + B_4(B_4B_1B_2B_3) = -B_4(B_1B_2B_3B_4).$$

Thus the pole of $[B_1B_2B_3]$ is the point B_4 and similarly with the other faces of the tetrahedron.

An anti-self-conjugate dyadic represents a null-system. For if,

$$BC = B_1C_1 + B_2C_2 + B_3C_3 + B_4C_4$$

is anti-self-conjugate,

$$\Sigma B_i C_i = -\Sigma C_i B_i = \tfrac{1}{2}\Sigma(B_i C_i - C_i B_i).$$

Hence

$$(\xi B)(C\xi) = \tfrac{1}{2}\Sigma\{(\xi B_i)(C_i\xi) - (\xi C_i)(B_i\xi)\} = 0.$$

For $(\xi B_i) = -(B_i\xi)$, $(\xi C_i) = -(C_i\xi)$ and since these products are numbers they are commutative. Since $(\xi B)(C\xi) = [\xi \cdot B(C\xi)]$, this shows that $B(C\xi)$ is a point in the plane ξ. Every plane ξ therefore passes by the correlation

$$X = B(C\xi)$$

into a point lying on ξ. The correlation is therefore a null-system.

As an operator on planes the anti-self-conjugate dyadic gives the same result as finding the poles of the planes with respect to the complex

$$p = \tfrac{1}{2}\{[B_1C_1] + [B_2C_2] + [B_3C_3] + [B_4C_4]\}$$

For

$$\begin{aligned}[p\xi] = \tfrac{1}{2}\Sigma[B_iC_i\xi] &= \tfrac{1}{2}\Sigma\{(\xi B_i)C_i - (\xi C_i)B_i\} = \\ &= \tfrac{1}{2}\{(\xi B)C - (\xi C)B\} \\ &= \tfrac{1}{2}\Sigma(B_iC_i - C_iB_i)\xi = \tfrac{1}{2}(BC - CB)\xi.\end{aligned}$$

This transformation therefore transforms each plane into its pole with respect to the complex p.

We have seen that any dyadic is the sum of two, one of which is self-conjugate representing a polarity, the other anti-self-conjugate representing a null system. The dyadic transforms any plane ξ into a point

$$B(C\xi) = \tfrac{1}{2}\{B(C\xi) + C(B\xi)\} + \tfrac{1}{2}\{B(C\xi) - C(B\xi)\}$$

on the line joining the points into which it is transformed by the polarity $\tfrac{1}{2}(BC + CB)$ and by the null-system $\tfrac{1}{2}(BC - CB)$.

With a one-one dyadic is associated a complex (or line)

$$p = [BC] = [B_1C_1) + [B_2C_2] + [B_3C_3] + [B_4C_4].$$

This is a covariant of BC as can be shown by the same argument used to show that the scalar of the one-three dyadic is an invariant. We shall call this the complex of the dyadic.

If BC transforms ξ into a point of η and η into a point on ξ, the intersection of ξ and η is a line of the complex p. For, if

$$(B(C\xi)\eta) = \Sigma(B_i\eta)(C_i\xi) = 0, \qquad \text{(A)}$$
$$(B(C\cdot\eta)\xi) = \Sigma(B_i\xi)(C_i\eta) = 0, \qquad \text{(B)}$$

then

$$(p\cdot\xi\eta) = \Sigma(B_iC_i\cdot\xi\eta) = \Sigma\{(B_i\xi)(C_i\eta) - (C_i\xi)(B_i\eta)\} = 0. \quad \text{(C)}$$

Conversely, through each line of the complex pass pairs of planes that are transformed in this way. For, if $[\xi\eta]$ is a line of the complex

and the transform of ξ is on η, (A) and (C) are satisfied and so (B) must be satisfied.

If
$$p = [BC] = \Sigma[B_iC_i] = 0, \tag{D}$$
the transform of η will always lie on ξ if that of ξ lies on η. This indicates a polarity. In fact, if $p = 0$, $(\mathrm{p}\xi) = 0$ and so

$$\tfrac{1}{2}(BC - CB)\xi = 0$$

where ξ is any plane, since the dyadic $\frac{1}{2}$ $(BC - CB)$ gives the same transformation as the null system set up by p. Then it is easy to see that

$$BC - CB = 0 \quad \text{or}$$
$$BC = CB.$$

The dyadic being self conjugate represents a polarity. In this case equation (D) shows that the four lines $[B_iC_i]$ belong to the same system of generators on a quadric. For it is a linear relation between four lines. *A polarity thus transforms the planes of any tetrahedron into the vertices of a second tetrahedron such that the lines joining corresponding vertices of the two tetrahedrons belong to the same set of generators on a quadric.*

In general two sets of four points B_i, and C_i can be found such that two one-one dyadics Φ and Ψ can be written in the form

$$\begin{aligned} \Phi &= B_1C_1 + B_2C_2 + B_3C_3 + B_4C_4, \\ \Psi &= \lambda_1B_1C_1 + \lambda_2B_2C_2 + \lambda_3B_3C_3 + \lambda_4B_4C_4, \end{aligned} \tag{36}$$

where the λ's are numbers. For, in general, there exists a set of four independent planes α_i which transform by both dyadics into the same set of four independent points B_i. If we take the C's as the vertices of the tetrahedron formed by the α's and properly choose the magnitudes of the B's, the dyadics will take the above forms.

If the planes of a tetrahedron C_1, C_2, C_3, C_4 transform into points B_1, B_2, B_3, B_4 such that the lines $[B_iC_i]$ belong to one system of generators of a quadric, the other system of generators of that quadric belong to p. For, if l is a generator of that second system, $(l\,B_iC_i) = 0$. Hence

$$(lp) = \Sigma(lB_iC_i) = 0,$$

and consequently l belongs to the complex p. We shall now show that such points B_i,C_i exist. We have just seen that two dyadics Φ, Ψ can be reduced to the form (36). This form shows that Φ

transforms the planes of the tetrahedron $C_1C_2C_3C_4$ into the points B_1, B_2, B_3, B_4. Now let Ψ represent a polarity. Then the lines $[B_iC_i]$ belong to a quadric. This shows that B_iC_i have the required property.

The three-three dyadic is the dual of the one-one. It transforms points into planes and can be expressed as a sum of four terms exactly dual to the one-one. Associated with each three-three dyadic

$$\alpha\beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4$$

is a covariant complex

$$p = [\alpha_1\beta_1] + [\alpha_2\beta_2] + [\alpha_3\beta_3) + [\alpha_4\beta_4].$$

As in case of the one-one dyadic it can be shown that if $\alpha\beta$ transforms a point A into a plane passing through B and the point B into a plane passing through A, then the line joining A and B is a line of the complex p.

13. **Two-three and two-one dyadics.** A two-three dyadic has the form

$$q\beta = \Sigma q_i\beta_i,$$

where the β's are planes and the q's are either lines or complexes. Since the planes can be expressed as linear functions of four not through a point, the dyadic can be written

$$q\beta = q_1\beta_1 + q_2\beta_2 + q_3\beta_3 + q_4\beta_4.$$

The consequents can be any four planes not passing through a point. The antecedents cannot, however, be assigned arbitrarily. For, the four complexes q_i in general have two lines l_1 and l_2 in common. We call these the singular lines of the dyadic. It is clear that $(l_1q)\beta = (l_2q)\beta = 0$.

As an operator on points the dyadic determines a transformation of points X into lines or complexes

$$p = q(\beta X) = \Sigma q_i(\beta_i X).$$

If the transform of X is a line,

$$(pp) = \Sigma(q_iq_k)(\beta_i X)(\beta_k X) = 0.$$

The points that are transformed into lines therefore lie on the quadric Q, whose equation is

$$\Sigma(q_i q_k)(\beta_i X)(\beta_k X) = 0.$$

Points on a generator of the quadric transform into lines any one of which is a linear function of any two others. Such a system of lines is a flat pencil. Hence the points of a generator transform into the lines of a flat pencil. If the quadric is not singular, take a skew quadrilateral on it and let $\beta_1, \beta_2, \beta_3, \beta_4$ be the planes determined by consecutive sides. Then q_1, q_2, q_3, q_4 will be lines for they correspond to the vertices of the quadrilateral (which are points of Q). Furthermore, q_1 and q_2 belong to a pencil and so intersect. Similarly q_2 intersects q_3 etc. The four lines therefore form a quadrilateral. If l is a diagonal of the quadrilateral it cuts all the lines q_i. Hence

$$(lq)\beta = 0.$$

The diagonals are therefore the same (being the singular lines of the dyadic) whatever quadrilateral $\beta_1, \beta_2, \beta_3, \beta_4$ is taken on the quadric Q. The flat pencils corresponding to points on a generator have their vertices on one of those diagonals and their planes pass through the other. All the generators of one system of Q give pencils with vertices on one diagonal, all those of the other system give pencils with vertices on the other diagonal.

Points of a plane transform into complexes with one system of generators on a quadric R in common. Points on the intersection of the plane with Q transform into the other system of generators of R. If the plane is tangent to Q, points on the intersection transform into lines of two plane pencils.

If Q degenerates into a cone, points on a generator still transform into the lines of a pencil. In case of the general quadric, the pencils corresponding to one system of generators have vertices on one line, those corresponding to the other system have vertices on another. In case of the cone, the two systems of generators coincide. Hence, the two lines on which the vertices lie, coincide. This line belongs to all the pencils and so is the transform of the vertex of the cone. In this case the four complexes q_1, q_2, q_3, q_4 have only one line in common.

Two points X, Y are harmonic with respect to Q if

$$\Sigma(q_i q_k)[(\beta_i X)(\beta_k Y) + (\beta_i Y)(\beta_k X)] = 0.$$

Since

$$\Sigma q_i \beta_i = \Sigma q_k \beta_k = \Phi = q\beta$$

this is equivalent to

$$[\Phi X\cdot\Phi Y] + [\Phi Y\cdot\Phi X] = 2[\Phi X\cdot\Phi Y] = 0.$$

This expresses that the complexes ΦX and ΦY that correspond to points harmonic with respect to the quadric Q are in involution. And conversely, if the complexes are in involution, the points are harmonic with respect to Q.

With a two-three dyadic

$$q\beta = \Sigma q_i\beta_i$$

is associated a point

$$P = \Sigma[q_i\beta_i].$$

This is a covariant of the dyadic. In order to obtain a geometrical interpretation for it suppose $\beta_1, \beta_2, \beta_3, \beta_4$ so chosen that the four points $[q_i\beta_i]$ lie in a plane. Then P (being a linear function of the points $[q_i\beta_i]$) lies in the same plane. That is, *if the vertices of a tetrahedron transform into four complexes such that the polar points of the opposite planes are four points in a plane, that plane passes through the point P.* That such tetrahedra $\beta_1, \beta_2, \beta_3, \beta_4$ exist can be shown as follows. Let

$$q\beta = q_1\beta_1 + q_2\beta_2 + q_3\beta_3 + q_3\beta_4.$$

If the points $[q_i\beta_i]$ satisfy a linear relation, the planes β_i have the required property. If not, at least one of the products $[q_i\beta_k]$, $i \neq k$, must be different from zero. Let $[q_4\beta_3]$ be different from zero. The dyadic can be written

$$q\beta = q_1\beta_1 + q_2\beta_2 + (q_3 - \lambda q_4)\beta_3 + q_4(\beta_4 + \lambda\beta_3).$$

Let α be the plane passing through $[q_1\beta_1]$, $[q_2\beta_2]$ and P. Since $[q_3\beta_3]$ and $[q_4\beta_4]$ are not equal $[q_4\beta_3]$ cannot equal both of them. Suppose $[q_3\beta_4]$ and $[q_4\beta_3]$ are different points. Then λ can be chosen such that $[q_1\beta_1]$, $[q_2\beta_2]$, $[(q_3 - \lambda q_4)B_3]$ and P lie in a plane. Since.

$$P = [q_1\beta_1] + [q_2\beta_2] + [q_3 - \lambda q_4)\beta_3] + [q_4(\beta_4 + \lambda\beta_3)]$$

this plane must pass through $[q_4(\beta_4 + \lambda\beta_3)]$. Hence $\beta_1, \beta_2, \beta_3, \beta_4 + \lambda\beta_3$ have the required property.

The discussion of the two-one dyadic can be taken by duality from the discussion of the two-three. The dyadic can be written in the form

$$qB = q_1B_1 + q_2B_2 + q_3B_3 + q_4B_4.$$

where the B's are any four independent points. The q's are not any four complexes but four that have two given lines in common. This dyadic transforms planes into lines or complexes. The planes which are transformed into lines envelope a quadric whose equation is

$$(q_i q_k)(B_i \alpha)(B_k \alpha) = 0.$$

Associated with the two-one dyadic is a covariant plane whose property is the dual of the covariant point of the two-three.

14. **Two-two dyadics.** A two-two dyadic has the form

$$rs = r_1 s_1 + r_2 s_2 + \ldots . r_n s_n,$$

the r's and s's being complexes (or lines). Since any complex can be expressed as a linear function of six that are linearly independent, the dyadic can be reduced to the form

$$rs = r_1 s_1 + r_2 s_2 + r_3 s_3 + r_4 s_4 + r_5 s_5 + r_6 s_6,$$

in which any six linearly independent complexes (or lines) can be taken as antecedents or consequents.

As an operator, the dyadic determines a transformation of complexes p into complexes

$$p' = r(sp) = \Sigma r_i(s_i p).$$

The lines that transform into lines belong to a quadratic complex g, consisting of lines p satisfying the equation

$$[r(sp)\cdot r(sp)] = 0.$$

Similarly, the complexes that transform into lines are the complexes p satisfying this quadratic equation.

The lines of the quadratic complex g transform into the lines of another quadratic complex g'. If two lines p and q of g intersect, the pencil of lines $p + \lambda q$ will, in general, transform into the pencil of complexes $p' + \lambda q'$. If, however, $p + \lambda q$ is the pencil of lines of g which lie in a singular [14] plane, $p' + \lambda q'$ will, for all values of λ, represent a line. Hence the lines in the singular planes of g transform into the lines in the singular planes of g'. It is to be noticed that the

14 In general the lines of a quadratic line complex which lie in a given plane envelope a conic. There are however ∞^2 planes in which these conics degenerate. These planes are called singular planes. See Jessop, page 89.

dyadic does not, in general, set up a contact transformation, i. e., intersecting lines do not, in general, go into intersecting lines. What we have shown is that the contact of pairs of lines in the singular pencils of g is preserved.

The lines of a pencil $p + \lambda q$, in general, transform into a pencil of complexes $p' + \lambda q'$. In this pencil are two special complexes that are lines. These correspond to the two lines of the pencil $p + \lambda q$ that belong to g. The lines of a plane transform into a two parameter linear system of complexes. The special complexes of this system are one system of generators on a quadric. These correspond to the lines of g which lie in the plane (and so envelope a conic). Similarly, the lines through a point go into a two parameter linear system of complexes whose quadric of singular elements corresponds to the cone of lines passing through the point and belonging to g.

Any two-two dyadic can be expressed as the sum of a self-conjugate and an anti-self-conjugate dyadic. For

$$rs = \Sigma r_i s_i = \tfrac{1}{2}\Sigma(r_i s_i + s_i r_i) + \tfrac{1}{2}\Sigma(r_i s_i - s_i r_i) = \tfrac{1}{2}(rs + sr) + \tfrac{1}{2}(rs - sr).$$

The first term is seen to be self-conjugate and the second term anti-self-conjugate.

A self-conjugate two-two dyadic is analogous to a polarity. For such a dyadic rs we have the relation

$$r(sp) = (pr)s.$$

Hence, if

$$(qr)(sp) = 0,$$

then

$$(pr)(sq) = 0,$$

that is, if p transforms into a complex (or line) in involution with q, then q transforms into a complex (or line) in involution with p.

Express rs in terms of six linearly independent complexes $q_1, q_2, \ldots q_6$,

$$rs = \Sigma a_{ik} q_i q_k.$$

If the dyadic is self-conjugate

$$\Sigma a_{ik} q_i q_k = \Sigma a_{ik} q_k q_i.$$

Hence

$$a_{ik} = a_{ki}.$$

Such a dyadic can always be reduced to the form

$$\lambda_1 p_1 p_1 + \lambda_2 p_2 p_2 + \ldots . + \lambda_6 p_6 p_6$$

the p's being complexes or lines and the λ's numbers. This reduction can be accomplished by the same transformation that reduces the quadratic form

$$\Sigma a_{ik} x_i x_k \tag{A}$$

to a sum of squares. Suppose, in fact, the transformation

$$y_i = \Sigma \mu_{ik} x_k \tag{B}$$

reduces (A) to the form

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots . + \lambda_6 y_6^2. \tag{C}$$

When we replace the y's in (C) by the values from (B) and expand, the coefficient of $x_i x_k$ will be

$$a_{ik} + a_{ki}$$

as in (A). Now let

$$p_i = \Sigma \mu_{ik} q_k \tag{D}$$

Then

$$\lambda_1 p_1 p_1 + \lambda_2 p_2 p_2 + \ldots . + \lambda_6 p_6 q_6 \tag{E}$$

will be equal to rs. For when we replace the p's in (E) by their values from (D) and expand, the sum of the coefficients of $p_i p_k$ and $p_k p_i$ in the result (as in the case of the quadratic form) will be

$$a_{ki} + a_{ik} = 2a_{ik}.$$

Also from the symmetry of (E) it is clear that the coefficients of $p_i p_k$ and $p_k p_i$ will be equal. Hence each is equal to a_{ik} and so (E) is equal to rs. In the manipulation of the dyadic and the quadratic form the principal difference is that $x_i x_k = x_k x_i$ while $p_i p_k$ and $p_k p_i$ need not be equal. The above discussion shows that the reduction to the sum of squares does not require that the individual products be commutative but merely that the whole dyadic be self conjugate.

Any self-conjugate two-two dyadic can be expressed in the form

$$rs = \lambda(p_1 p_4 + p_4 p_1) + \mu(p_2 p_5 + p_5 p_2) + \nu(p_3 p_6 + p_6 p_3)$$

where $p_1, p_2 \ldots p_6$ are complexes (or lines) and λ, μ, ν numbers. To show this, first reduce rs to the form

$$rs = \lambda_1 q_1 q_1 + \lambda_2 q_2 q_2 + \ldots . + \lambda_6 q_6 q_6.$$

Then let

$$\begin{aligned}
p_1 &= \sqrt{\lambda_1}q_1 + \sqrt{-\lambda_4}q_4 \\
p_4 &= \sqrt{\lambda_1}q_1 - \sqrt{-\lambda_4}q_4 \\
p_2 &= \sqrt{\lambda_2}q_2 + \sqrt{-\lambda_5}q_5 \\
p_5 &= \sqrt{\lambda_2}q_2 - \sqrt{-\lambda_5}q_5 \\
p_3 &= \sqrt{\lambda_3}q_3 + \sqrt{-\lambda_6}q_6 \\
p_6 &= \sqrt{\lambda_3}q_3 - \sqrt{-\lambda_6}q_6
\end{aligned}$$

Using these values it is readily seen that

$$rs = \tfrac{1}{2}(p_1p_4 + p_4p_1) + \tfrac{1}{2}(p_2p_5 + p_5p_2) + \tfrac{1}{2}(p_3p_6 + p_6p_3).$$

If none of the quantities $q_1, q_2, \ldots, q_6$ are zero this has the form required. If p_1 is zero we can replace it by any complex and let λ be zero in the expression

$$rs = \lambda(p_1p_4 + p_4p_1) + \ldots.$$

Hence every self-conjugate two-two dyadic can be reduced to this form, $p_1, p_2, \ldots p_6$ being complexes.

Every self-conjugate two-two dyadic can be expressed as the product of a dyadic and its conjugate. That is, if Ψ is any such dyadic, a dyadic $\Phi = rs$ can be found such that

$$\Phi_c\Phi = \Psi.$$

To show this reduce Ψ to the form

$$\Psi = \lambda(p_1p_4 + p_4p_1) + \mu(p_2p_5 + p_5p_2) + \nu(p_3p_6 + p_6p_3). \qquad \text{(A)}$$

Let $q_1, q_2 \ldots q_6$ be the edges of a tetrahedron, q_1 and q_4, q_2 and q_5, q_3 and q_6 being the pairs of non-intersecting edges. Choose the magnitudes of the q's such that

$$[q_1q_4] = [q_2q_5) = [q_3q_6] = 1.$$

Now let

$$\Phi = \mu_1q_1p_1 + \mu_2q_2p_2 + \ldots . \mu_6q_6p_6.$$

Then

$$\Phi_c\Phi = \mu_1\mu_4(p_1p_4 + p_4p_1) + \mu_2\mu_5(p_2p_5 + p_5p_2) + \mu_3\mu_6(p_3p_6 + p_6p_3).$$

Comparison of this with (A) shows that we can make $\Phi_c\Phi = \Psi$ by choosing $\mu_1, \mu_2 \ldots \mu_6$ such that

$$\mu_1\mu_4 = \lambda, \qquad \mu_2\mu_5 = \mu, \qquad \mu_3\mu_6 = \nu.$$

By using the theorem just proved we can show that *the complex g of lines which are transformed into lines is a general quadratic complex,*

i. e. any quadratic complex consists of the lines thus transformed by some two-two dyadic. In fact g consists of all lines p satisfying the equation

$$[\Phi p \cdot \Phi p] = [p\Phi_c \cdot \Phi p] = [p\Psi p] = 0.$$

We have just shown that Φ can be determined such that Ψ is any self-conjugate two-two dyadic. Now any homogeneous quadratic equation in the Plücker coordinates of a line p can be written in the form

$$[p\,\Psi\,p] = 0$$

where Ψ *is a self-conjugate two-two dyadic.* Hence *any quadratic complex is the complex g of some two-two dyadic* Φ.

Suppose next, that Φ is an anti-self-conjugate two-two dyadic. Let $q_1, q_2 \ldots . q_6$ be six linearly independent complexes. Then Φ can be written

$$\Phi = \Sigma a_{ik} q_i q_k.$$

Since

$$\Phi = -\Phi_c$$

$$\Sigma a_{ik} q_i q_k = -\Sigma a_{ik} q_k q_i$$

whence

$$a_{ik} = -a_{ki}.$$

The dyadic can therefore be written

$$\Phi = a_{12}(q_1q_2 - q_2q_1) + a_{13}(q_1q_3 - q_3q_1) + a_{14}(q_1q_4 - q_4q_1) + \ldots$$

Suppose one of the coefficients a_{ik}, say a_{14}, is not zero. Let

$$\lambda_{14}p_4 = a_{12}q_2 + a_{13}q_3 + a_{14}q_4 + a_{15}q_5 + a_{16}q_6,$$

λ_{14} being an arbitrarily assigned number which is zero if the right side of the equation is zero. By using this equation eliminate q_4 from Φ and so reduce it to the form

$$\Phi = \lambda_{14}(q_1p_4 - p_4q_1) + \ldots .$$

In this form q_1 appears only in the term $q_1p_4 - q_4p_1$. The others contain q_2, q_3, q_4, q_5, q_6. Similarly, if q_2 occurs in more than one of the combinations $q_i\,q_k - q_kq_i$ a new complex can be introduced such that q_2 will appear in only one combination. Finally q_3 can be treated in the same way. The dyadic will then have the form

$$\Phi = \lambda_{14}(q_1p_4 - p_4q_1) + \lambda_{25}(q_2p_5 - p_5q_2) + \lambda_{36}(q_3p_6 - p_6q_3) + \ldots$$

Each of the remaining terms contains p_4, p_5, or p_6. A sum of terms

$$\lambda_{14}(q_1p_4 - q_4p_1) + \lambda_{24}(q_2p_4 - q_4p_2) + \ldots.$$

can be combined into a single term

$$\lambda_{14}(p_1p_4 - p_4p_1)$$

where

$$\lambda_{14}p_1 = \lambda_{14}q_1 + \lambda_{24}q_2 + \ldots..$$

Hence the dyadic can be reduced to the form

$$\Phi = \lambda_{14}(p_1p_4 - p_4p_1) + \lambda_{25}(p_2p_5 - p_5p_2) + \lambda_{36}(p_3p_6 - p_6p_3).$$

The complexes $p_1, p_2 \ldots p_6$ can be taken linearly independent. If, for example, p_6 were a linear function of the others it could be replaced by this linear function and the dyadic would then be expressed in terms of 5 complexes, $p_1, p_2 \ldots . p_5$. The above process would then reduce Φ to two terms instead of three. This is a special case in which one of the coefficients $\lambda_{14}, \lambda_{25}, \lambda_{36}$ is zero.

An anti-self-conjugate dyadic Φ transforms any complex p into a complex in involution with p. For

$$\Phi p = p\Phi_c = - p\Phi.$$

Hence

$$[p\Phi p] = - [p\Phi p] = 0.$$

This expresses that Φp and p are in involution.

A very important type of two-two dyadic is that which gives the same transformation of lines and complexes as a point collineation or a point-plane correlation. The peculiarity of such a transformation is that it preserves contact, that is, it transforms intersecting lines into intersecting lines and complexes in involution into complexes in involution. If Φ is such a dyadic

$$\Phi p_1 \cdot \Phi p_2 = p_1\Phi_c\Phi p_2 = 0$$

whenever

$$p_1p_2 = 0.$$

These are linear equations in the coordinates of p_1 and p_2 such that the first is always satisfied when the second is. *Hence there must be a constant λ such that*

$$p_1\Phi_c\Phi p_2 = \lambda p_1p_2.$$

If I *is the two-two idemfactor this is equivalent to*

$$p_1[\Phi_c\Phi - \lambda I]p_2 = 0.$$

Hence

$$\Phi_c\Phi = \lambda I.$$

Conversely, if this condition is satisfied the dyadic determines a collineation or correlation.

If a collineation or correlation is set up by either a self-conjugate or an anti-self-conjugate dyadic, the transformation is an involution. For then

$$\Phi = \pm \Phi_c$$
$$\Phi_c\Phi = \lambda I.$$

Hence

$$\Phi^2 = \pm \lambda I$$

which shows that two applications of the transformation Φ gives identity. Hence Φ determines an involution.

II. Double Products.

15. The double product [15] of two dyads AB and CD is defined as

$$[AC]\ [BD].$$

In general this is a new dyad. If one of the factors $[AC]$, $[BD]$ is a scalar, it is however an extensive quantity. If both factors are scalars, the double product is a scalar.

The double product of two dyadics

$$AB = \Sigma A_i B_i$$
$$CD = \Sigma C_i D_i$$

is the sum of terms

$$AB : CD = \Sigma[A_i C_k]\ [B_i D_k] = [AC]\ [BD]$$

obtained by multiplying the two dyadics distributively. Since the products $[A_i\, C_k]$ and $[B_i\, D_k]$ are distributive, if the antecedents or the consequents of either dyadic are replaced by their values as linear

15 Gibbs-Wilson, Vector Analysis, page 306. Wilson's paper quoted above.

functions of other extensive quantities; the double product of the dyadics in the new form will be equal to that in the old. The double product is thus independent of the form in which the dyadics are expressed. Hence it is a covariant of the two dyadics.

The geometrical interpretation of the double product $[AC]\,[BD]$ depends on whether $[BD]$, that is $[B_iD_k]$, is a progressive or a regressive product. Suppose $[BD]$ is progressive and X a space complementary to $[BD]$. Express X as a product of planes ξ_i,

$$X = [\xi_1\xi_2\ldots\xi_m].$$

Divide the planes into two sets. Call the product of the planes in one set α_i and the product of the planes in the other set β_i and arrange the planes in the sets so that

$$X = [\alpha_i\beta_i].$$

If the sets are so chosen that β_i is complementary to D, the reduction formulas (§4) give

$$[DX] = \Sigma\alpha_i(D\beta_i),$$

the summation being for all combinations of the planes, $\xi_1\xi_2\ldots\xi_m$ in sets α_i, β_i. Therefore

$$[AC]\,[BDX] = [AC]\,[B\cdot DX] = [AC]\Sigma(B\alpha_i)\,(D\beta_i).$$

This result can be written

$$[AC]\,[BDX] = \Sigma[A(B\alpha_i)C(D\beta_i)].$$

This shows that if α_i is transformed by AB and β_i by CD and if the transforms are then joined, $[AC]\,[BDX]$ will be a linear function of the joins. This is true in whatever way X is expressed as a product of planes.

If $[BD]$ is regressive we proceed as before except that X is expressed as a product of points instead of planes. If B and D are of complementary dimensions, either points or planes may be used.

Suppose, for example,

$$AB = \Sigma A_iB_i, \qquad CD = \Sigma C_iD_i$$

A_i, B_i, C_i, D_i being points. Let L be any line and ξ_1, ξ_2, two planes through it. Transform ξ_1, by AB and ξ_2 by CD. The result is two points whose join is a line p. Transform ξ_2 by AB and ξ_1 by CD. Let the join of the corresponding points be q. Then $[AC]\,[BD]$ transforms

L into a line or complex r such that r is a linear function of p and q. Hence p and q are polar with respect to r (see §5). This is true for every pair of planes ξ_1 and ξ_2 passing through L. *To find the complex or line into which* $[AC]$ $[BD]$ *transforms* L, *we therefore find the complex or line with respect to which* p *and* q *are polar whatever pair of planes are taken through* L.

As a second case consider a correlation AB and collineation $C\gamma, A_i$, B_i, C_i being points and γ a plane. The double product

$$[AC]\ (B\gamma)$$

is a complex or line. In this case $(B\gamma)$ is a number and so X must be a number. If P_1, P_2, P_3, P_4 are the four vertices of a tetrahedron we may take

$$X = (P_1P_2P_3P_4).$$

Transform P_1 by the collineation $C\gamma$ and the opposite plane $[P_2P_3P_4]$ by AB. Let the join of the resulting points be p_1. Proceed in the same way with the other vertices of the tetrahedron and the planes opposite them. *The above general discussion shows that* $[AC](B\gamma)X$, *and so,* $[AC](B\gamma)$ *is a complex or line belonging to the congruence determined by the four lines thus obtained. This is true for every tetrahedron* P_1, P_2, P_3, P_4.

As a final illustration consider the case of a one-one dyadic AB and a two-two dyadic pq, The double product

$$[Ap]\ [Bq]$$

is a three-three dyadic. Let X be any point and ξ_1, ξ_2, ξ_3 three planes through it. Let

$$R_1 = A(B\xi_1), \qquad r_1 = p(q_1\xi_2\xi_3).$$

The join of R_1 and r_1 is a plane $[R_1r_1]$. Permuting ξ_1, ξ_2, ξ_3 we get three such planes. The three planes intersect in a point. *By the above general discussion that point is on the plane into which* $[Ap]$ $[Bq]$ *transforms* X. *This is true for every set of three planes through* X.

16. **Double products with idemfactors.** In particular the double product of a dyadic Φ and an idemfactor is an invariant or covariant of Φ. This is a special case of the preceding general discussion. For example let $A\alpha$ be the dyadic determining the identical point collineation and let

$$\Phi = B\beta$$

represent any other point collineation. Then the double product

$$[AB]\ [\alpha\beta]$$

transforms any line L into a line or complex

$$L' = [AB](\alpha\beta\cdot L) = -[BA](\alpha\cdot\beta L).$$

Since $(\alpha\beta L)$ is a pure regressive product. Furthermore, this is equivalent to

$$-B\cdot A\alpha\cdot[BL] = -[B\cdot\beta L] = [BL\cdot\beta]$$

since $A\alpha$ is the idemfactor. This complex is determined as follows. Let X, Y be two points on L and X', Y' their transforms by Φ. Join X to Y' and Y to X'. Then by the general theorem of the preceding section L' is a linear function of the two lines thus obtained. That is, these lines are polar lines with respect to L'. This is true whatever pair of points X, Y are taken on L. This proves the following geometrical theorem. Let X, Y, Z be any three distinct points on L and $X', Y' Z'$, three distinct points on any other line. A dyadic $B\beta$ can be found which will transform X, Y, Z into X', Y', Z'. Therefore the three pairs of lines XY', YX'; XZ', ZX'; YZ', ZY' are pairs of polar lines with respect to a complex, namely, the complex into which $[AB]\ [\alpha\beta]$ transforms $[XY]$. This is the generalization of the theorem of Pappus for the hexagon inscribed in two lines in a plane.

The dyadic $[AB]\ [\alpha\beta]$ will represent a collineation if and only if every line XY transforms into a line $X'Y'$ cutting it. The collineation $B\beta$ then gives a transformation of lines which is a null system. By §14 $B\beta$ then determines an involution.

We can write $B\beta$ in the form

$$B\beta = B[CDE] = B_1[C_1D_1E_1] + B_2[C_2D_2E_2] + B_3[C_3D_3E_3] + B_4[C_4D_4E_4].$$

Then

$$\begin{aligned}[AB]\ [\alpha\beta] &= [AB]\ [\alpha\cdot CDE].\\ &= [AB]\{(\alpha E)[CD] - (\alpha D)[CE] + (\alpha C)[DE]\}\\ &= -B\cdot A\alpha\cdot\{E[CD] - D[CE] + C[DE]\}\\ &= -[BE]\ [CD] + [BD]\ [CE] - [BC]\ [DE].\end{aligned}$$

This gives the dyadic in a form that does not involve the idemfactor. The result can be obtained from

$$B[CDE]$$

by a symbolic multiplication of B and CDE analogous to the outer product.

Again, let pq be the two-two idemfactor. Then

$$[Bp]\,[\alpha q]$$

is a three-three dyadic which transforms any plane ξ into a plane

$$\xi' = [Bp]\,(\beta q\cdot\xi) = B\cdot pq\cdot\beta\xi = [B\cdot\beta\xi]$$

To interpret this let X, Y, Z be any three non-collinear points on ξ and X', Y', Z' their transforms by $B\beta$. Join X' to YZ, Y' to XZ, Z' to XY. By the general theorem of the preceding section these planes intersect in a point on ξ'. This is true, whatever points X, Y, Z are taken on ξ. This involves the following geometrical theorem. Take four points X, Y, Z, W in a plane, no three of which are collinear, and four points X', Y', Z', W' in a second plane. A dyadic $B\beta$ can be found which will transform X, Y, Z, W, into X', Y', Z', W'. Joining the points X, Y, Z and X', Y', Z', W' as above we get a point. Similarly, Y, Z, W and Y', Z', W' determine a second point etc. In this way we get four points which lie in a plane, namely, in the plane into which $[Bp]\,[\beta q]$ transforms $[XYZ]$.

If we write $B\beta$ in the form

$$B\beta = B[CDE],$$

$$\begin{aligned}[Bp]\,[\cdot\beta q] &= [Bp]\{(qCD)E - (qCE)D + (qDE)C\}\\ &= B\cdot pq\cdot\{[CD]E - [CE]D + [DE]C\}\\ &= [BCD]E - [BCE]D + (BDE)C.\end{aligned}$$

This can be regarded as a symbolic product of B and $[CDE]$ analogous to the outer product.

Finally let αA be the three-one idemfactor. The double product of this and the dyadic $B\beta$ is

$$(b\alpha)\,(\beta A) = -\,b\cdot\alpha A\cdot\beta = -\,(b\beta).$$

When this is zero the scalar, or linear invariant, of the dyadic vanishes. When the double product of two dyadics vanishes, we shall call them *apolar*. To interpret this let A_1, A_2, A_3, A_4 be the vertices of a tetrahedron and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the opposite planes. Let $B\beta$ transform A_1, A_2, A_3, A_4 into the points A_1'. A_2', A_3', A_4'. The general theorem states that $(A_1'\alpha_1)$, $(A_2'\alpha_2)$, $(A_3'\alpha_3)$, $(A_4'\alpha_4)$ will satisfy a linear relation.

If then three of these numbers are zero the fourth will be zero also. This is Pasch's theorem [16] that if a collineation represented by $B\beta$, with scalar invariant zero, transforms each of three vertices of a tetrahedron into a point of the opposite plane, it will transform the fourth vertex into a point of its opposite plane.

Let $A\alpha$ and $B\beta$ both be idemfactors and let $L = (XY)$ be any line. Then

$$\begin{aligned}[AB](\alpha\beta\cdot L) &= [AB](\alpha\beta\cdot XY)\\ &= [AB]\{(\alpha X)(\beta Y) - (\alpha Y)(\beta X)\}\\ &= A(\alpha X)\cdot B\beta\cdot Y + B(\beta X)\cdot A\alpha\cdot Y\\ &= [XY] + [XY] = 2L.\end{aligned}$$

Hence

$$[AB](\alpha\beta\cdot L) = 2L$$

and so

$$[AB]\,[\alpha\beta] = 2pq$$

where pq is the two-two idemfactor. *The double product of the one-three idemfactor with itself is thus twice the two-two idemfactor.*

The double product of the idemfactor $A\alpha$ and a one-one dyadic CD is the line or complex

$$[CA]\,[D\alpha] = -\,C\cdot A\alpha\cdot D = -\,[CD].$$

We have called this the complex of the dyadic. The general theorem states that if CD transforms the planes of the tetrahedron X_1, X_2, X_3, X_4 into the points Y_1, Y_2, Y_3, Y_4, the complex $[CD]$ is a linear function of the four lines $[X_1Y_1]$, $[X_2Y_2]$, $[X_3Y_3]$, $[X_4Y_4]$, that is, the two lines cutting these four lines belong to the complex $[CD]$.

The double product of the three-one idemfactor αA and a dyadic CD is

$$(C\alpha)[DA] = DC\cdot\alpha A = [DC] = -\,[CD].$$

The interpretation of this coincides with that given in §12.

If $[CD]$ is zero, the one-one dyadic CD represents a polarity. Thus the condition that a one-one dyadic represent a polarity *is that it be apolar with the one-three or three-one idemfactor.*

The double product of CD and the two-two idemfactor pq is the three-three dyadic

$$\Phi = [Cp]\,[Dq].$$

---

16 Loc. cit.

As an operator this determines a point plane correlation,

$$\xi = \Phi X = [Cp]\,[DqX] = C\cdot pq\cdot DX = [CDX].$$

This is the correlation which transforms each point X into its polar plane with respect to the complex $[CD]$. If CD represents a polarity $[CD]$ is zero and so the double product of CD and pq is zero. Hence *the condition that a correlation represent a polarity is that it be apolar with the two-two idemfactor.*

Let rs be any two-two dyadic. Symbolically we may write this

$$rs = [CD]\,[EF].$$

The double product of rs and the idemfactor $A\alpha$ is

$$\begin{aligned}[rA]\,[s\alpha] &= [CDA]\,[EF\cdot\alpha] \\ &= [CDA]\,[(E\alpha)F - (F\alpha)E] \\ &= CD\cdot A\alpha\cdot\{FE - EF\} \\ &= CD\cdot\{FE - EF\} \\ &= [CDF]E - [CDE]F.\end{aligned}$$

This result has the same form as the product $[CD\cdot EF]$ where C, D, E, F, are points in a plane. The dyadic determines a collineation which transforms a plane ξ into the plane

$$\eta = [rA](s\alpha\xi) = -\,[r\cdot s\xi].$$

To interpret this collineation geometrically let rs transform the sides $[A_2A_3]$, $[A_3A_1]$, $[A_1A_2]$ of a triangle in ξ into the complexes p_1, p_2, p_3 respectively. Let

$$\alpha_1 = [A_1p_1]$$

be the polar plane of A, with respect to p_1 and similarly let

$$\alpha_2 = [A_2p_2],\quad \alpha_3 = [A_3p_3].$$

The general theorem states that $\alpha_1, \alpha_2, \alpha_3$, will intersect on η. The fact that this is true whatever points A_1, A_2, A_3, A_4 are taken on ξ proves a geometrical theorem. Suppose for example rs represents a collineation. By such a collineation four non collinear points A_i of one plane could be transformed into any four non-collinear points B_i of another plane. Join A_1 to $B_2\,B_3$, A_2 to $B_3\,B_1$, and A_3 to $B_1\,B_2$. The three planes intersect in a point. Similarly A_2, A_3, A_4 and

B_2, B_3, B_4 determine a second point, etc. The four points thus obtained lie in a plane.

The double product of the one-two dyadic $B[CD]$ and the three-one idemfactor $A\alpha$ is the two-one dyadic

$$\begin{aligned}\Phi &= [BA]\,[CD\cdot\alpha] \\ &= [BA]\{(C\alpha)D - (D\alpha)C\} \\ &= B\cdot A\alpha\cdot\{DC - CD\} \\ &= [BD]C - [BC]D.\end{aligned}$$

To interpret this let $B[CD]$ transform the lines YZ, ZX, XY of a plane ξ into the points X', Y', Z' respectively. Then Φ will transform the plane ξ into a complex

$$[BD](C\xi) - [BC](D\xi)$$

which is a linear function of the three lines XX', YY', ZZ'. Hence the complex contains the quadric of these lines. If we take four points X, Y, Z, W of ξ four quadrics will thus be determined which all belong to the same complex. In this case the lines of the four point will transform into points of a four line in another plane. Since the dyadic can be so determined that this four line is arbitrary, this shows that if the corresponding triangles of a four point in one plane and a four line in another plane are joined as above the four quadrics determined will belong to a complex.

If

$$[BCD] = 0,$$
$$[CD](B\xi) + [DB](C\xi) + [BC](D\xi) = 0$$

for every plane ξ. Hence

$$[CD]B = [BD]C - [BC]D$$

which shows that in that case the dyadic Φ is the conjugate of $B[CD]$.

The double product of $B[CD]$ and αA is the plane of the dyadic

$$(B\alpha)[CDA] = [BCD].$$

Let p_1, p_2, p_3 be three lines intersecting in a point of this plane. If p_3 transforms into a point of the plane $p_1\,p_2$ and p_2 into a point of p_3, p_1, the general theorem shows that p_1 will transform into a point of the plane p_2p_3. If $[BCD] = 0$ this will be true of any three lines intersecting in a point.

The double product of $B[CD]$ and the two-two idemfactor pq is

$$[Bp][CDq] = [BCD].$$

17. **Dyadics apolar to all the idemfactors.** If the double product of rs and $A\alpha$ is zero, we may consider rs as analogous to a polarity. In that case, if

$$\begin{aligned} rs &= [CD][EF] \\ [rA][\alpha s] &= [CDF]E - [CDE]F = 0 \end{aligned}$$

and hence

$$[CDF]E = [DCE]F, \tag{A}$$

Let X, Y be any two points. Then by direct expansion we get

$$\begin{aligned} (XYEF)[CD] = (XYCD)[EF] &+ (CDEF)[XY] \\ - (XCDE)[YF] &- (YCDF)[XE) \\ + (XCDF)[YE] &+ (YCDE)[XF]. \end{aligned}$$

But from (A) we have

$$(CDFE) = (CDEF) = 0,$$

Also

$$\begin{aligned} (XCDF)[YE] &= (XCDE)[YF], \\ (YCDF)[XE] &= (YCDE)[XF]. \end{aligned}$$

Hence

$$(XYEF)[CD] = (XYCD)[EF].$$

Since this is true for all value of X and Y

$$[EF][CD] = [CD][EF].$$

The dyadic is therefore self conjugate and its scalar vanishes. Conversely, if these conditions are satisfied it is easy to show that rs is apolar to the one-three, the three-one and the two-two idemfactors. It is thus apolar to all the idemfactors.

The double product of two polarities is apolar to all the idemfactors. For let CD and $C'D'$ be two polarities. Then

$$[CD] = [C'D'] = 0.$$

Hence, if $A\alpha$ is the one-three idemfactor

$$\begin{aligned} [CC'\cdot A][DD'\cdot\alpha] &= [CC'A]\{(D\alpha)D' - (D'\alpha)D\} \\ &= -[CC'D]D' + [CC'D']D = 0, \end{aligned}$$

which shows that $[CC'] [DD']$ is apolar to $A\alpha$. It follows that it is also apolar to αA and pq.

We have already noted that a one-one dyadic which represents a polarity is apolar to all the idemfactors. The same is true of a three-three dyadic.

If rs is apolar to the idemfactors, the double product of rs and a one-one polarity CD is apolar to the idemfactors. For

$$[Cr \cdot Ds] = [(Cr \cdot s)D] - (Cr \cdot D)S = 0$$

Similarly we can show that the double product of rs and a three-three polarity is apolar to the idemfactors.

If rs is apolar to the two-two idemfactor pq, its scalar

$$(rp)(sq) = (rs) = 0.$$

In this case the general theorem states that if rs transforms each of five edges of a tetrahedron into a complex to which the opposite edge belongs, the same will be true of the sixth.

We have thus shown that *if any dyadic is apolar to the one-three or three-one idemfactor, it is apolar to all the idemfactors and, excepting the case of the two-two apolar to the two-two idemfactor, if a dyadic is apolar to any idemfactor it is apolar to all. Furthermore, if two dyadics are apolar to all the idemfactors their double product (if it is a dyadic) is also.*

18. **Dyadics symbolically derived from a given dyadic.** If we write a one-three dyadic symbolically in the form

$$B[CDE]$$

we have seen that its double product with the one-three and the two-two idemfactors are

$$-[BC][DE] + [BD][CE] - [BE][CD]$$

and

$$[BCD]E - [BCE]D + [BDE]C$$

respectively. These can be considered as obtained by a sort of symbolic multiplication of B and CDE analogous (except for a possible change of sign) to the outer multiplication.

Similarly, from any dyadic a series of dyadics are obtained. These are all double products of the original dyadic with the various idemfactors.

In a somewhat similar way from a line $[BC]$ we get a dyadic

$$CB - BC.$$

This is the product

$$\begin{aligned} A[\alpha\cdot BC] &= A\{(\alpha C)B - (\alpha B)C\} \\ &= A\alpha\cdot C\cdot B - A\alpha\cdot B\cdot C = CB - BC. \end{aligned}$$

This shows that if

$$[BC] = [DE]$$

then

$$BC - CB = DE - ED.$$

In this discussion $[BC]$ and $[DE]$ may be complex lines or simple lines. If $[BC]$ is a complex, the dyadic $BC - CB$ gives the plane-point polar transformation with respect to the complex.

Similarly, from a plane $[BCD]$ we get two dyadics

$$A[\alpha\cdot BCD] = B[CD] - C[BD] + D[BC]$$

and

$$[BCD\cdot\alpha]A = [CD]B - [BD]C + [BC]D.$$

The first of these as an operator on lines gives the point in which the line cuts the plane. The second as an operator on planes gives the line in which $[BCD]$ cuts the plane.

Those same dyadics are also obtained by multiplying the plane $[BCD]$ with the idemfactor pq.

In the same way by considering a point as the product of three planes, two dyadics can be determined.

19. **Double products of dyadic with themselves.** The double product of a one-three dyadic

$$B\beta = B'\beta'$$

with itself is a two-two dyadic

$$[BB'][\beta\beta']$$

which gives the transformation of lines determined by the transformation $X' = b(\beta X)$ of points. For, if $b\beta$ transforms X, Y into X', Y', then $[BB'][\beta\beta']$ transforms the line $[XY]$ into a linear function of $[X'Y']$ and $[Y'X']$, that is, into the line $[X'Y']$.

Similarly, the triple product

$$[BB'B''][\beta\beta'\beta'']$$

of $B\beta$ with itself represents the same transformation as an operator on planes. For, if $B\beta$ transforms three points X, Y, Z into X', Y', Z', then

$$[BB'B''] [\beta\beta'\beta''] = B\beta : [B'B''] [\beta'\beta'']$$

transforms the plane $[XYZ]$ into a linear function of the planes $[X'Y'Z']$, $[Y'Z'X']$, etc., that is into the plane $[X'Y'Z']$.

In the same way it is easily shown that the double and the triple product of a one-one dyadic with itself determine the same transformation in lines and in points.

The double product of a one-two dyadic with itself is zero. For, if $Ap = A'p'$,

$$[AA'](pp') = [A'A](p'p) = - [AA'](pp').$$

Since the double product is equal to its negative, it is zero. The same is true of the double product of a two-three dyadic with itself.

20. **Hamilton-Cayley equations.** It has been shown by various writers that a one-three or a three-one dyadic in three dimensions satisfies an algebraic equation of the fourth degree called the Hamilton-Cayley equation [17] of the dyadic.

That the two-two dyadic satisfies an equation of the sixth degree might be inferred from the fact that the transformation set up by a two-two dyadic in three dimensions can be interpreted as a transformation of points in a space of five dimensions. In general, there will then be six linearly independent complexes left invariant by the dyadic. Taking these as prefactors, the dyadic can be written

$$\Phi = rs = \lambda_1 p_1 q_1 + \lambda_2 p_2 q_2 + \ldots + \lambda_6 p_6 q_6.$$

Since

$$r(sp_1) = \mu p_1,$$

where μ is constant, it follows that

$$(p_1 q_2) = (p_1 q_3) = \ldots = (p_1 q_6) = 0.$$

Thus each p is in involution with all the q's except the one associated with it. If the p's and q's are lines, this is the configuration called a double six. In general we may call it a double six of complexes.

17 Whitehead's Universal Algebra, page 261. Bôcher's Introduction to Higher Algebra, Chapter XXII.

If I is the two-two idemfactor, it is easily seen that

$$\{\Phi - \lambda_1(p_1q_1)I\}\cdot p_1 = 0.$$

Similarly,

$$\{\Phi - \lambda_2(p_2q_2)I\}\cdot p_2 = 0,$$

etc. Consider the product

$$\Psi = \{\Phi - \lambda_1(p_1q_1)I\}\{\Phi - \lambda_2(p_2q_2)I\}\dots\{\Phi - \lambda_6(p_6q_6)I\}.$$

It is clear that

$$\Psi p_6 = 0.$$

Since Ψ and I are commutative, any factor could be put last. Hence

$$\Psi p_1 = \Psi p_2 = \dots = \Psi p_6 = 0.$$

Since $p_1 \dots p_6$ on assumed linearly indepedent,

$$\Psi \equiv 0.$$

When expanded this has the form

$$A_0\Phi^6 + A_1\Phi^5 + \dots + A_3 - \Phi + A_6 I = 0.$$

This is the Hamilton-Cayley equation satisfied by the dyadic.

That a polynomial equation is satisfied by any dyadic whose prefactors and postfactors on dual may be shown in the following way. Let R_i and S_k be dual. Let $R_i \dots R_n$ be linearly independent and let $S_1 \dots S_n$ be linearly independent. Suppose

$$\Phi = \Sigma A_{ik} R_i S_k,$$
$$\Psi = \Sigma b_{ik} R_i S_k$$

are commutative. Then

$$\Phi\Psi - \Psi\Phi = \Sigma b_{ik}(\Phi R_i)S_k - A_{ik}(\Psi R_i)S_k = 0.$$

Since the S'^s are linearly independent, the coefficient of S_k in this equation is zero. That is,

$$b_{1k}(\Phi R_1) - a_{1k}(\Phi R_1) + \dots + b_{nk}(\Phi Rn) - A_{nk}(\Psi R_n)$$
$$= (b_{1k}\Phi - A_{1k}\Psi)\cdot R_1 + \dots + (b_{nk}\Phi - A_{nk}\Psi)\cdot R_n = 0.$$

There on n such equations. Eliminating $R_2 \ldots R_n$ as in solving algebraic equations, we get

$$0 = \Delta \cdot R_1 = \begin{vmatrix} b_{11}\Phi - a_{11}\Psi, & b_{21}\Phi - a_{21}\Psi, \ldots, & b_{n1}\Phi - a_{n1}\Psi \\ b_{12}\Phi - a_{12}\Psi, & b_{22}\Phi - a_{22}\Psi, \ldots, & b_{n2}\Phi - a_{n2}\Psi \\ \cdots & \cdots & \cdots \\ b_{1n}\Phi - a_{1n}\Psi, & \cdots\cdots & b_{nn}\Phi - a_{nn}\Psi \end{vmatrix}.$$

Similarly,

$$\Delta \cdot R_2 = \Delta \cdot R_3 = \ldots = \Delta \cdot R_n = 0.$$

Since $R_1, \ldots R_n$ on linearly independent,

$$\Delta = 0.$$

when expanded this has the form

$$A_0\Phi^n + A_1\Phi^{n-1}\Psi + \ldots + A_{n-1}\Phi\Psi^{n-1} + A_n\Psi^n = 0.$$

The coefficient A_0 is the discriminant of Ψ, and A_n that of Ψ. Hence, if either of the dyadics is non-singular, the coefficients do not all vanish. In particular, if Ψ is an idemfactor, the equation becomes

$$A_0\Phi^n + A_1\Phi^{n-1} + \ldots + A_{n-1}\Phi + A_n I = 0.$$

This is the Hamilton-Cayley equation satisfied by Φ.